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AN ECONOMIC ANALYSIS OF
PRICE ADJUSTING SAMPLING PLANS

A THESIS


Presented to
The Faculty of the Division of Graduate
Studies and Research
by
Thomas Henry Lynch, Jr.

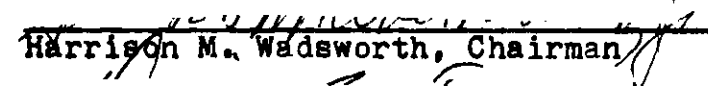
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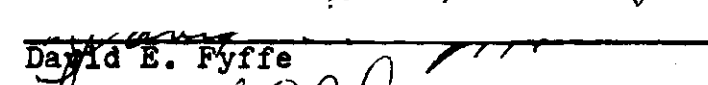
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AN ECONOMIC ANALYSIS OF
PRICE ADJUSTING SAMPLING PLANS

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Dedicated to

The Woofer

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SUMMARY

The purpose of this research was to conduct an economic analysis of a relatively new concept in quality control, price adjusting sampling plans. This concept eliminates the traditional idea of rejecting bad lots and substitutes therefore a scheme whereby a certain price is paid for each good item and nothing is paid for a defective item. The price paid for the total lot is based on the estimated lot fraction defective and a consumer indifference curve. As the lot fraction defective increases, the amount the consumer pays for the lot decreases in such a fashion that he is indifferent to the quality of the lot.

This thesis concludes that although existing price adjusting plans are single sample plans, it is not possible to derive economic benefits from the application of sequential analysis to the problem of determining how large a sample should be taken and how much should be paid for the lot since the basic problem is one of parameter estimation and not hypothesis testing. Sequential analysis is simply not applicable to this type of problem.

To reassure the risk-averse producer and/or consumer who might be reluctant to employ a price adjusting scheme, this research produced a technique for determining an interval for the most likely value for the actual price paid for each good unit. The actual price paid per good unit will vary depending on how accurately the estimated lot fraction defective approximates the true lot fraction defective.

Because there is a risk of economic loss to the producer and

consumer associated with parameter estimation, loss functions for both parties have been developed. These functions identify the magnitude and likelihood of a loss. Additionally, should a plan be selected which protects the producer against a predetermined loss, the economic impact of this plan on the consumer is analyzed.

By varying the slope of the consumer indifference curve, this research shows that the consumer can increase both the incentive for good quality and the penalty for bad.

Finally, some practical suggestions are offered regarding the implementation of a price adjusting sampling plan.

CHAPTER I

INTRODUCTION

Traditionally, the exact price of a lot, or batch, of manufactured goods is the result of an agreement between the producer and the consumer reached most often prior to the actual shipment of the goods. At some point prior to accepting the lot, the consumer will most likely perform some type of acceptance inspection. If the quality is good, the lot will be accepted; otherwise, it will be rejected. When a lot is rejected, the producer's profits will be diminished by the additional shipping and manufacturing costs necessary to improve the quality of the lot. Involved are round trip transportation costs, unpacking and repacking, and the actual cost of identifying and reworking the defective items. Ultimately, these costs must be passed on to the consumer if the producer is to remain in business.

All of the additional costs that result from the rejection of a lot can be avoided if instead of rejecting lots of poor quality, all lots are accepted at a variable price depending on the lot fraction defective. This concept, introduced by Foster (12), is based upon the assumption that the consumer can develop an indifference curve which will relate the price he is willing to pay for a particular lot to the quality of that lot, where quality is measured in terms of lot fraction defective.

Unit price is divided into two categories; P dollars are paid for each good item and zero dollars are paid for each defective item. Because the actual number of good units in a particular lot is not known

a priori, that quantity is estimated based upon the number of defectives found in the sample inspected. The precise number of defectives could be determined by inspecting the entire lot but in most cases that would be prohibitively expensive.

The purpose of this thesis is to conduct an economic analysis of the price adjusting sampling plans developed by Foster. First, since Foster's plans are single sampling plans, intuition suggests that savings could accrue from using sequential sampling techniques to reduce the average number of items sampled. This thesis will show that item-by-item sequential sampling is not appropriate for use with Foster's price adjusting plans.

Second, this thesis will develop limits for the most likely value of the actual price paid by the consumer for each good item. Economic loss functions will also be developed which will indicate to the producer and the consumer the magnitude of a potential loss and the risk associated therewith. A related technique will be developed whereby the consumer can reduce, though not absolutely minimize, his losses.

CHAPTER II

LITERATURE SURVEY

From 1929, when the first acceptance sampling plan was published by H. F. Dodge and H. G. Romig of Bell Telephone Laboratories, until the present, acceptance sampling has been used to determine whether a lot of manufactured goods should be accepted or rejected. A typical attributes sampling plan stipulates that the consumer randomly select a sample of size n from the lot in question and accept the lot if the number of defectives is less than or equal to some number $c < n$ and reject the lot if the number of defective items is greater than c . All of the sampling plans developed since 1929, double, multiple, sequential, chain sampling, etc. have been developed to assist in making the decision to accept or reject a lot of goods. The differences among the plans are the ability to discriminate between good and bad quality, the sample size necessary to accomplish that discrimination, and the ease of administration.

During World War II, the concept of acceptance sampling was extended to include incentives which would encourage the production of good quality. Although the Army Ordnance Sampling Tables, developed in 1942, did not provide for the payment of economic rewards to the producer when he submitted lots of good quality, they did penalize the producer when lots of substandard quality were submitted. Specifically, the producer was assured that so long as he maintained the lot fraction defective at or below a prescribed level, the Acceptable Quality Level

or AQL, his lots would be accepted a high percentage of the time. However, should the fraction defective exceed the specified limit by a statistically significant amount, the incoming lots would be subjected to more stringent inspection, resulting in a higher rejection rate. Conversely, if the quality was consistently better than the AQL by a significant amount, the inspection criteria would be reduced; i.e., it would be more permissive. The rules for switching back and forth among the three plans, normal, tightened and reduced, provided an incentive to the vendor to insure that the quality of lots produced was maintained at better than the acceptable level.

Only within about the last ten years has any thought been given to using the results of acceptance sampling as a basis for determining an economic reward to be paid directly to the producer. Hill (15) in 1960, discusses the idea of developing economic incentives which would encourage the production of good quality. Certainly, the prospect of having a lot of bad quality rejected is in itself an incentive, but Hill points out that acceptance sampling does little to change the quality of goods received by the consumer. The level of quality depends directly on the producer's process curve, or the prior distribution of lot fraction defective, and to change that curve to reflect a higher level of quality requires that the producer be offered some economic incentive to do so. In his discussion, Hill illustrates his point using a single sampling plan, but he does not discuss the effect that the Military Standard 105 series of acceptance sampling plans has on the producer's process curve. The MIL-STD-105 plans evolved from the World War II Army Ordnance Tables, Navy Tables of the same vintage, and efforts of an

American-British-Canadian Working Group that sought to derive a common standard for the three countries.

In 1964 Flehinger and Miller (11) developed a scheme for incentive contracts based on the assumption that some specific level of quality will be optimum for both the consumer and the producer. If the agreed upon level of quality is met, as determined by an acceptance test, then a premium is paid to the producer.

The approach used by Flehinger and Miller is the simultaneous minimization of the producer's and the consumer's loss functions which are dependent upon the fraction defective and the particular acceptance test parameters. The premium paid to the producer is determined by negotiation; the minimum value is the amount the producer must invest to achieve the optimal level of quality. The maximum value is the amount the consumer profits by getting improved quality. This arrangement permits both producer and consumer to profit, assuming, of course, that the cost to the producer of improved quality is less than the benefit to the consumer.

In 1966, Durbin (10) developed a pricing scheme somewhat similar to that proposed by Flehinger and Miller. The basic difference is that with Durbin's plan instead of the consumer paying a premium if quality is good, the consumer pays a unit price $\phi(x)$ based on x , the number of defectives found in a single sample of size n . As with the Flehinger and Miller approach, expressions are developed for consumer and producer profit. Durbin describes the situation as a constrained, two person non-zero-sum game where the producer can control p the fraction defective and the consumer can prescribe n the sample size. It is assumed that the

producer can state the minimum unit price, w , which he desires assurance of receiving if true quality is as good as p^* . Since even with acceptable quality ($p \leq p^*$) any batch may yield a sample containing a disproportionately high number of defectives, the producer will accept a contingent pricing policy which given n and $p \leq p^*$, promises payment less than w with frequency less than some specified α . Similarly, it is assumed that the consumer can state v , the maximum unit price he is willing to pay when quality is as poor as $p_b > p^*$ and β , the relative frequency with which overpayment will be permissible.

The model is formulated as a linear programming problem where the consumer seeks to maximize his expected profit by choosing n and $q(x)$ knowing that the producer will choose the fraction defective, p , that will maximize his profit. The constraints are the upper and lower bounds on the pricing policy described above.

In 1967, Roeloffs (16) recognized that most sampling plans penalize the producer when process quality decreases from an acceptable level, but do not reward the producer for improving the quality beyond the acceptable level. He points out that the economic benefit of quality improvement beyond the acceptable level accrues to the consumer, while the cost of such improvement must be borne by the producer. The producer has no incentive to expend resources to improve the process beyond the acceptable level, but the consumer, on the other hand, would benefit from process improvement beyond the acceptable level. Roeloffs proposes a Price Differential Sampling Plan (PDSP) which would adjust the price paid for a lot according to the fraction defective observed in the sample drawn from the lot.

As with most single sampling plans, the PDSP identifies in the usual manner a sample size n and an acceptance number c . Roeloffs then develops a schedule of prices (in terms of price per unit) to be paid for the lot assuming that the number of defectives in the lot does not exceed c , the acceptance number.

A typical schedule of prices is as shown below,

Defectives in Sample	Action
0	Accept lot; unit price = q_0
1	Accept lot; unit price = q_1
i	Accept lot; unit price = q_i
c	Accept lot; unit price = q_c
$> c$	Reject lot

The value of q_0 is determined by the consumer and is the maximum price he would be willing to pay for entirely defect-free product. The values of q_i , where $i = 1, 2, \dots, c$, are based on a linear decreasing function of i and have values such that if quality merely remains at the acceptable level, the expected price will be the price that would be paid under a conventional fixed price acceptance sampling plan.

In 1970 Foster (12) introduced the concept of price adjusting sampling plans (PASP), which eliminate the costly rejection process. All lots are accepted and are priced based upon the number of defectives found in the sample and the consumer indifference curve. The first step in Foster's plan is to establish through negotiation a value for P , the price per good unit. Then the consumer must develop his indifference

function which in the simplest form might be,

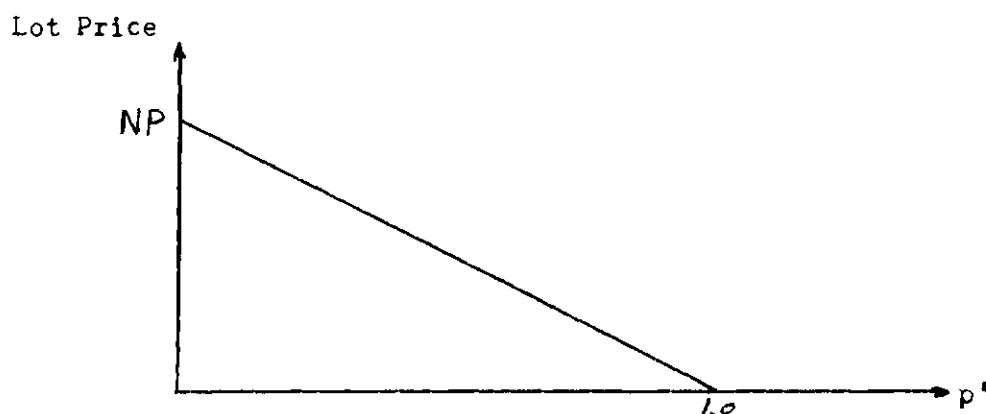


Figure 1. Consumer Indifference Curve.

where p' is the true lot fraction defective and N is the lot size. To protect himself against a severe loss that might result if a small sample was used to estimate p' , the producer stipulates that if the true lot fraction defective is at some value p_1 , the probability that the price per non-defective unit be less than some lower limit L , should be less than or equal to some predetermined value α . This stipulation prescribes a value for n which can be evaluated since Foster assumes that the ratio n/N is sufficiently small such that the binomial distribution closely approximates the hypergeometric distribution. After the sample is inspected and x , the number of defectives, is identified, the total price paid for the lot, based on the indifference curve shown in Figure 1 is

$$P_{\text{Total}} = NP(1 - x/n)$$

In 1972 Foster and Perry (13) developed the concept of a price adjusting single sampling (PASS) plan which in essence is a more detailed discussion of Foster's earlier work. The second paper includes tables

which greatly reduce the difficulty of determining the proper sample size.

Paralleling the evolution of economic incentives as an integral part of an acceptance sampling plan was the development of interest in item-by-item sequential sampling. This concept was first devised by Abraham Wald in April, 1943 while he was associated with the Statistical Research Group, Columbia University and was initially described in a classified government report entitled, "Sequential Analysis of Statistical Data: Theory." This report was submitted by the Statistical Research Group to the Applied Mathematics Panel, National Defense Research Committee, September, 1943. Subsequently, the material was declassified and published by Wald in June, 1945, in the Annals of Mathematical Statistics, and again in September, 1945 as a series of reports entitled "Sequential Analysis of Statistical Data," published by Columbia University. Finally, in 1947, Wald published his classic text, entitled Sequential Analysis.

To quote Wald (19), "By a sequential test of a statistical hypothesis is meant any statistical test procedure which gives a specific rule, at any stage of the experiment (at the n^{th} trial for each integral value of n), for making one of the following three decisions: (1) to accept the hypothesis being tested (null hypothesis), (2) to reject the null hypothesis, (3) to continue the experiment by making an additional observation. Thus, such a test procedure is carried out sequentially. On the basis of the first trial, one of the three decisions mentioned above is made. If the first or the second decision is made, the process is terminated. If the third decision is made, a second trial is performed. Again on the basis of the first two trials one of the three decisions is made and if the third decision is reached a third trial is performed,

etc. This process is continued until either the first or the second decision is made."

In summary, there has been a trend over the past thirty years away from the idea of merely penalizing the producer for bad quality and toward the concept of encouraging good quality through the use of monetary rewards. Foster introduced the idea of accepting all lots at a price dependent on the number of defectives found in the sample inspected. However, no attempt has been made to date to link the merits of a plan such as Foster's with the economies that accrue from the use of sequential analysis.

CHAPTER III

REVIEW OF PASP AND ITS RELATIONSHIP TO SEQUENTIAL ANALYSIS

The purpose of this chapter is to demonstrate that although the conventional item-by-item sequential sampling plan developed by Wald is much more efficient than a conventional single sampling plan, a parallel situation does not exist when one tries to develop a sequential sampling plan with price adjusting. Intuition would lead one to believe that a sequential sampling plan could be developed and that the expected number of observations would be much less than the number required for a price adjusting single sampling (PASS) plan. Such is not the case.

The Concept of Price Adjusting Sampling Plans (PASP)

As developed by Foster (12), the concept of price adjusting sampling plans (PASP) is unique in that it eliminates the usual purpose of sampling inspection, namely, to determine if a lot should be accepted or rejected. All lots are accepted with the price paid for the lot being a function of x , the number of defectives found in a sample of size n .

As was mentioned earlier, the total price paid for a lot of N items will depend upon the quality of the sample. If the sample contains no defectives, a maximum price, NP , will be paid for the lot where P is a price per good unit. P dollars are paid for each good unit and zero dollars for each defective. The value P is a fixed amount and is agreed upon by the producer and the consumer during contract negotiations.

If the quality of the lot is not perfect, that is, the true lot fraction defective p' is greater than zero, then the price paid for the lot should be reduced by some amount $Ng(x)$. The expression for total price of the lot then becomes,

$$(1) \quad P_{\text{Total}} = N[P - g(x)]$$

In order for such a plan to work, the consumer must be indifferent to quality. That is, he must pay an amount for each lot such that as quality varies, from $p' = 0$ to $p' = 1.0$, he is not concerned because the price paid for the lot decreases in a manner acceptable to him.

An ideal consumer indifference curve for a lot of size N might look like that shown in Figure 2.

Lot Price



Figure 2. Consumer's Ideal Indifference Curve.

The interpretation of this curve is as follows. If the quality is perfect, he is willing to pay NP for the lot. However, as p' increases, the problem of screening out the defectives changes from an

occasional annoyance to a slowdown in production. The price paid for a bad lot will have to be considerably less than the price paid for a good lot in order to offset the problems encountered in the manufacturing process resulting from having received a large number of defective items, especially if the defectives are not identified until late in the manufacturing process.

The producer, however, takes a different viewpoint. In order to protect his profit margin, and indeed the survival of his enterprise, he would like to sell his product using the price adjusting scheme only if the indifference curve is of the form shown in Figure 3.

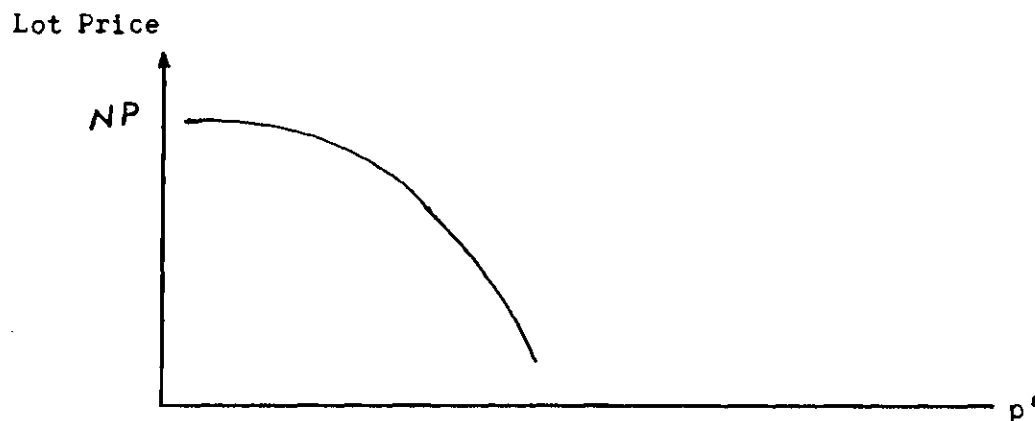


Figure 3. Producer's Ideal Indifference Curve.

Since neither of the two curves shown in Figures 2 and 3 is acceptable to both parties, a compromise will be made. A linear indifference curve of the form shown below will be used in the following discussion.

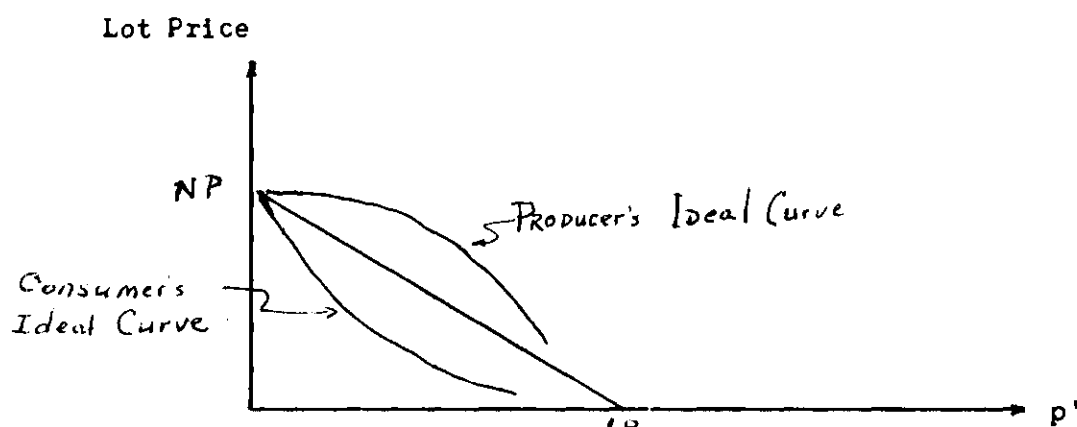


Figure 4. Linear Indifference Curve.

The meaning of this curve is that the consumer, for example is indifferent to quality on the condition that the lot price is determined by the relationship,

$$(2) \quad \text{LOT PRICE} = NP(1 - p')$$

This means that as p' increases Lot Price decreases linearly until $p' = 1.0$.

The linear indifference curve could be of the form

$$\text{LOT PRICE} = \begin{cases} NP(1 - p'/a) & 0 \leq p' < a \\ 0 & \text{otherwise} \end{cases}$$

which means that the consumer is unwilling to pay anything for the lot when $p' \geq a$, but this will be treated as a separate problem in Chapter V.

Referring to Equation (1), we see that P_{Total} is an expression for lot price that is dependent upon x the number of defectives in the sample. To review the discussion found in Foster and Perry (13), the expression for P_{Total} should be acceptable to the consumer if the expected

value of the amount the consumer will pay, P_{Total} , is equal to LOT PRICE, the indifference curve relating dollars to the true lot fraction defective. That is, if over the long run the consumer can expect that the amount he gets based on the results of sampling will equal the amount that he would like to receive, LOT PRICE. That is, if

$$(3) \quad E[P_{\text{Total}}] = \text{LOT PRICE}$$

then the consumer will be indifferent to the quality he receives.

Thus,

$$(4) \quad E[P_{\text{Total}}] = E[NP - Ng(x)] = NP(1 - p')$$

$$NP - NE[g(x)] = NP - NP(p')$$

$$E[g(x)] = P(p')$$

Assuming that $g(x)$ is linear and of the form

$$g(x) = dx \quad \text{where } d \text{ is a constant}$$

we find that

$$(5) \quad E[g(x)] = dE[x] = dnp'$$

substituting (5) into (4) yields

$$dnp' = P(p')$$

or,

$$d = P/n$$

Since the expression for $g(x) = dx$ has been evaluated as

$$g(x) = P/n \cdot x$$

Equation (1) becomes,

$$(6) \quad P_{\text{Total}} = N[P - g(x)] = N[P - P/n \cdot x]$$

Factoring out P , (6) becomes

$$(7) \quad P_{\text{Total}} = NP[1 - x/n]$$

Protecting the Producer

In order for the pricing scheme outlined in the preceding section to be acceptable to the producer, he must have certain assurances that p represents a good estimate of p' , the true lot fraction defective. If $\hat{p} > p'$, the producer will sell the lot for less than it is worth, but if $\hat{p} < p'$, he will reap additional profit. His main concern is to guard against a loss. Hence, once P , the theoretical price per good unit is identified, the actual price per good unit should be determined.

$$\begin{aligned} \text{Actual cost per good unit} &= \frac{\text{Total price paid for the lot}}{\text{Total No. of good units actually in the lot}} \\ &= \frac{NP(1 - x/n)}{N(1 - p')} \\ &= P\left(\frac{1 - x/n}{1 - p'}\right) \end{aligned}$$

Call this expression C , that is,

$$(8) \quad C = P\left(\frac{1 - x/n}{1 - p'}\right)$$

Since C is a random variable whose actual value depends on how accurately x/n estimates p' , the producer will want to specify a lower limit, L for C when quality is good, i.e., $p' \leq p_1$. Such a specification will reduce the magnitude and frequency of a loss which would result when a random sample contains a disproportionately high number of defectives. The expression which represents producer protection is,

$$(9) \quad \Pr\{C < L | p' \leq p_1\} \leq \alpha$$

where α is some small value such as 0.05 or 0.10.

Example

The following example illustrates the application of a PASS plan.

Suppose that in the past the consumer has been paying \$0.20 per item and that each lot has had an average per cent defective of 4%. To arrive at a value for P , the price per good item, we write

$$\begin{aligned} P &= \frac{\text{Total price paid for each lot in the past}}{\text{Total actual good items in each lot}} \\ &= \frac{N(0.20)}{N(1 - 0.04)} = \$0.2083 \end{aligned}$$

Let us assume that the producer wants to be very sure that he doesn't suffer a severe loss by adopting this type of plan so he chooses a value of $L = \$0.18$ that is only 10% less than he used to receive for items that were defective 4% of the time. He wants to be assured of getting at least \$0.18, 99% of the time. Referring to Equation (9) this means that

$$L = \$0.18 \quad \text{and}$$

$$\alpha = 0.01$$

Substituting Equation (8) into (9) gives

$$(10) \quad \Pr \left\{ P\left(\frac{1 - x/n}{1 - p'}\right) \leq L \mid p' \leq p_1 \right\} \leq \alpha$$

Equation (10) may be transformed by algebraic manipulation and by the substitution of p_1 for p' to

$$(11) \quad \Pr \left\{ x/n \geq \left[1 - \frac{L}{p}(1 - p_1)\right] \right\} \leq \alpha$$

Substituting the values of the parameters yields

$$\Pr \left\{ x/n \geq 0.171 \right\} \leq 0.01$$

Referring to the tables of the cumulative binomial probability distribution, for $p' = 0.04$ we see that a sample size of $n = 18$ is the smallest sample which will yield the desired protection.

For $n = 18$,

$$\Pr \left\{ x \geq (18)(0.171) \right\} = \Pr \left\{ x \geq 3.08 \right\}$$

Because x is discrete, this is equivalent to

$$\Pr \left\{ x \geq 4 \right\} = 0.005$$

Since 0.005 is smaller than 0.01, the producer is satisfied. The consumer is agreeable to this plan because he agreed to the indifference curve which determined the payment schedule.

To summarize the plan, it stipulates that a sample of 18 items be inspected, the number of defectives x , counted, and a payment of $NP(1 - x/n)$ be made to the producer.

Protecting the Consumer

Not only is the producer concerned about the economic impact of arriving at a value of x/n which is a bad estimate of p' , the true lot fraction defective, but the consumer has similar worries. If $x/n < p'$, then the actual price per good item that he pays may be intolerable to him. Since C the actual cost per good unit is a random variable which depends on how accurately x/n estimates p' , we might visualize the probability distribution function of C to look something like the following,

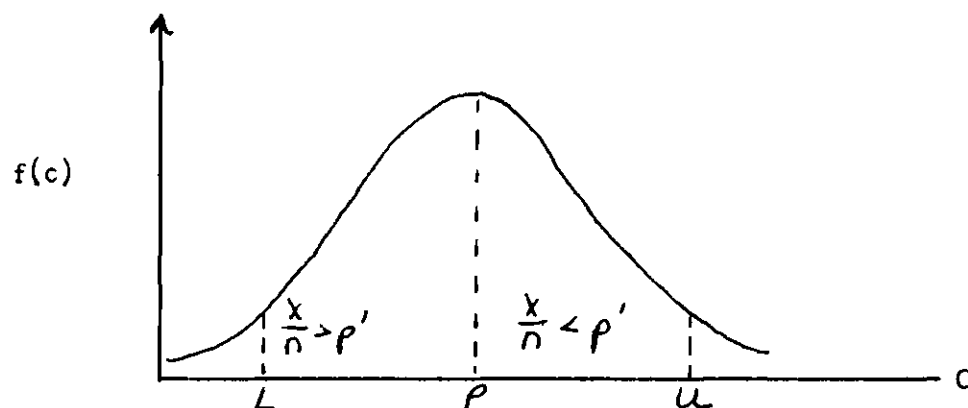


Figure 5. Probability Distribution Function of Variable Cost Per Good Unit.

The expected value of C is P , which occurs when $x/n = p'$; when $x/n > p'$, C tends toward L and perhaps beyond. When $x/n < p'$, then the price the consumer will have to pay tends toward U .

Conceivably, the consumer and the producer might agree to develop a PASP plan to protect the consumer's interests. The consumer, for example, may have very reluctantly consented to a linear indifference curve rather than one of the type shown in Figure 2. To compensate for this concession the consumer may insist on high assurance that if quality

is bad, ($p' = p_2 > p_1$), he won't be required to pay any more than some upper limit, U , ($U > P$) for each good unit actually in the lot. Specifically, his requirement is as follows,

$$(12) \quad \Pr \left\{ C < U \mid p' \geq p_2 \right\} \geq 1 - \beta$$

where β is some small value specified by the consumer.

Substituting for C in Equation (8), Equation (12) becomes

$$\Pr \left\{ P \left(\frac{1 - x/n}{1 - p'} \right) < U \mid p' \geq p_2 \right\} \geq 1 - \beta$$

Setting p' equal to p_2 and rearranging terms, this becomes,

$$(13) \quad \Pr \left\{ x/n > \left[1 - \frac{U}{P} (1 - p_2) \right] \right\} \geq 1 - \beta$$

Mostly likely, the values of U , β , and p_2 chosen by the consumer will not result in the same value of n as would be generated by a producer protection plan as discussed in the previous section. The reason for this is that the plan to protect the consumer and the plan to protect the producer are two separate plans. The parameters U , L , p_1 , p_2 , α and β are not common to both plans. One is concerned with x/n being greater than some quantity a low percentage of the time, whereas the other is concerned with x/n being greater than some other quantity a high percentage of the time.

Example

To illustrate, the consumer protection plan, the example used to demonstrate a producer protection plan will be used. That is, $P = \$0.2083$. Assume further that the consumer specifies,

$$U = \$0.225$$

$$\beta = 0.05$$

$$p_2 = 0.08$$

Evaluating Equation (13) using the above values leads to,

$$\Pr \left\{ x/n > \left[1 - \frac{0.225}{0.2083} (1 - 0.08) \right] \right\} \geq 1 - 0.05$$

or

$$\Pr \left\{ x/n > 0.0064 \right\} \geq 0.95$$

From the tables of the cumulative binomial probability distribution, a minimum value of n is found to be $n = 36$. Actually,

$$\Pr \left\{ x \geq (36)(0.0064) \right\} = 0.95030$$

The plan functions in a manner similar to the producer protection scheme, that is, take a sample of 36, count the number of defectives x and pay an amount equal to $NP(1 - x/n)$ for the lot.

Wald's Sequential Analysis and Price Adjusting Sampling

The purpose of conventional single sampling plans is to answer the following questions -- for a given set of criteria, should a lot be accepted or rejected? The criteria are usually specified in the following manner,

$$(14) \quad \begin{aligned} \Pr \left\{ \text{Accept the lot} \mid p' = p_1 \right\} &\geq 1 - \alpha \\ \Pr \left\{ \text{Accept the lot} \mid p' = p_2 \right\} &\leq \beta \end{aligned}$$

Where p_1 and p_2 represent good and bad quality respectively and α and β have small values and $p_1 < p_2$.

"Accept the lot" means mathematically that x the number of defectives in a sample of size n is smaller than some value c . When p_1 , p_2 , α and β have been specified, n and c can be evaluated using the two expressions in (14). A conventional single sampling plan then works as follows. Take a sample of n items and count the number of defectives. If $x \leq c$ accept the lot; if $x > c$ reject it.

Wald's item-by-item sequential sampling plan satisfies the requirements stipulated in (14) using a device that Wald terms a Sequential Probability Ratio (SPR). The meaning of the SPR can be more readily understood if the requirements of (14) are expressed in a more general, graphical form. The curve shown in Figure 6 is known as an operating characteristic (OC) curve.

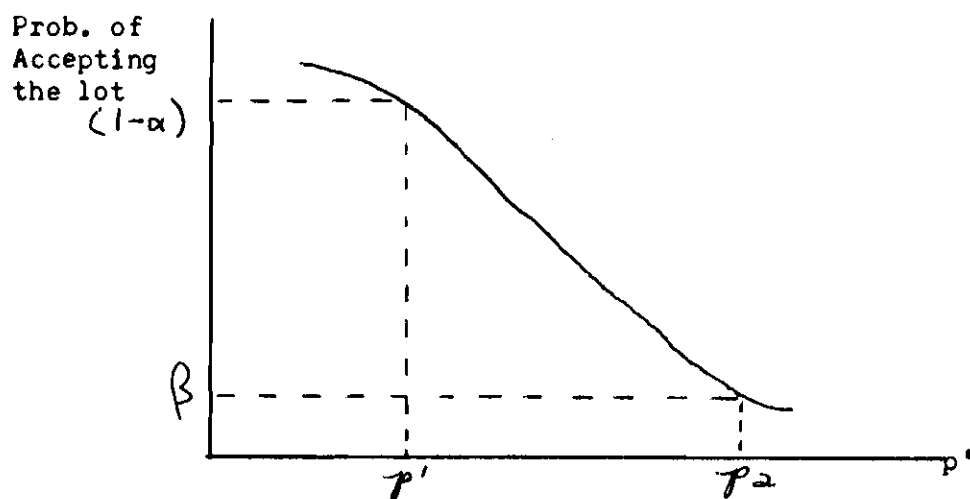


Figure 6. Operating Characteristic Curve

This curve indicates that as p' , the true fraction defective, varies from zero to one, the probability of accepting the hypothesis

that the lot is good (and hence of accepting the lot) varies from one to zero. The two points specified by (14) determined the shape of the curve.

The SPR is developed as follows. A null hypothesis H_0 is established, and for an acceptance sampling plan this means the quality of the lot is good, i.e., $p' = p_1$. The alternate hypothesis, H_a is that quality is bad, i.e., $p' = p_2$. To simplify the discussion, the SPR shall be defined as follows,

$$(15) \quad \text{SPR} = \frac{\text{Pr (2)}}{\text{Pr (1)}}$$

where,

Pr (2) = probability of getting the results that have thus far been obtained, i.e., the specific number of defectives found after the n^{th} trial, if H_a the alternate hypothesis is true.

If it can be assumed that x is binomially distributed, then

Pr (2) can be expressed as

$$\text{Pr (2)} = p_2^x (1 - p_2)^{n-x}$$

Pr (1) = Probability of getting the results that have thus far been obtained, i.e., the specific number of defectives found after the n^{th} trial, if H_0 the null hypothesis is true.

If it can be assumed that x is binomially distributed, then Pr (1) can be expressed as

$$\text{Pr (1)} = p_1^x (1 - p_1)^{n-x}$$

Wald places boundaries on the SPR, and once the boundaries are

exceeded as a result of a certain set, (x, n) , then sampling stops because a decision has been reached. The decision rules are:

Reject the lot if $SPR > A$

Accept the lot if $SPR < B$

where

$$A = \frac{\text{Probability of rejection if } H_a \text{ is true}}{\text{Probability of rejection if } H_0 \text{ is true}}$$

and

$$B = \frac{\text{Probability of acceptance if } H_a \text{ is true}}{\text{Probability of acceptance if } H_0 \text{ is true}}$$

Thus the boundaries on the SPR become

$$(16) \quad \frac{\beta}{1 - \alpha} \leq SPR \leq \frac{1 - \beta}{\alpha}$$

This relationship can be more readily visualized if Figure 6 is redrawn with some added notation,

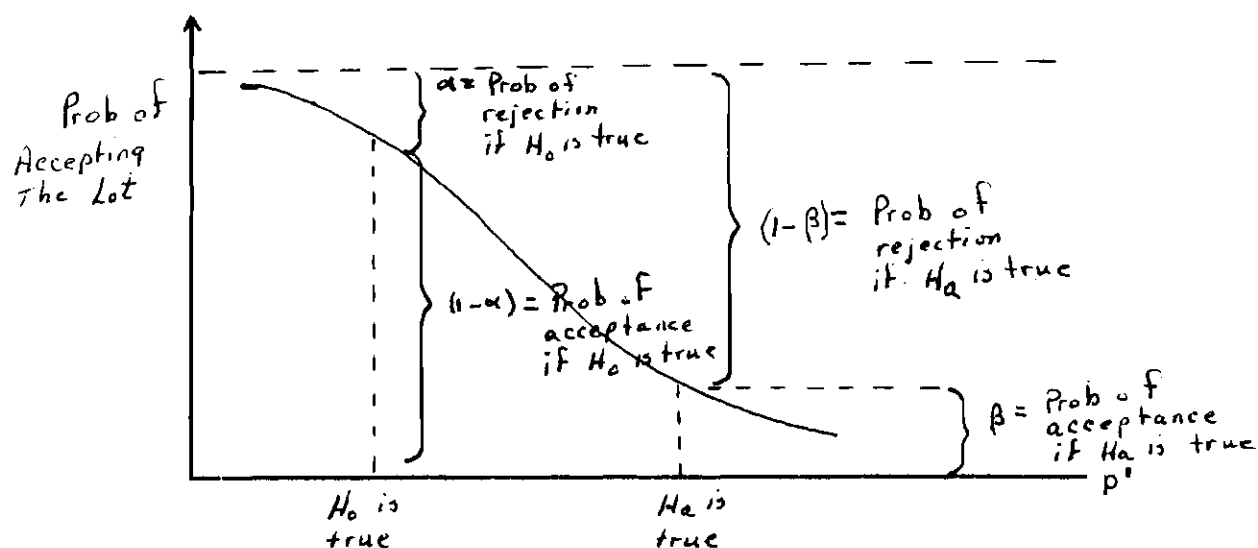


Figure 7. Operating Characteristic Curve.

Wald (19) indicates that for the values of α and β most frequently employed the sequential test results in an average saving of at least 47% in the necessary number of observations compared with a single sampling test.

Because such significant savings are possible with sequential analysis, it is intuitively appealing to try to apply this same approach to Foster's price adjusting sampling plans. To evaluate the merit of such an idea, consider first the problem of developing a plan which protects the producer.

To apply sequential analysis, two points on an operating characteristic curve must be defined. Recalling the earlier discussion of Foster's producer protection plan, we note that only one point was used namely a point defined as follows.

$$\Pr \left\{ C < L \mid p' \leq p_1 \right\} \leq \alpha$$

The point is defined by (p_1, α) . The question that concerns the producer is the way the probability that $C > L$ varies as p' varies. This is analogous in conventional sampling plans to the way the probability of accepting the lot varies as p' varies. Hence, a cost characteristic curve can be developed which would be based on the following hypotheses. $H_0 : p' = p_1$, $H_a : p' = p_2$. The curve would appear as shown in Figure 8.

We note that the second point defined by (p_2, θ) does not involve the consumer and his interest in having $C < U$. The consumer's interest in being assured that C does not exceed the upper limit U would be treated as an entirely separate problem.

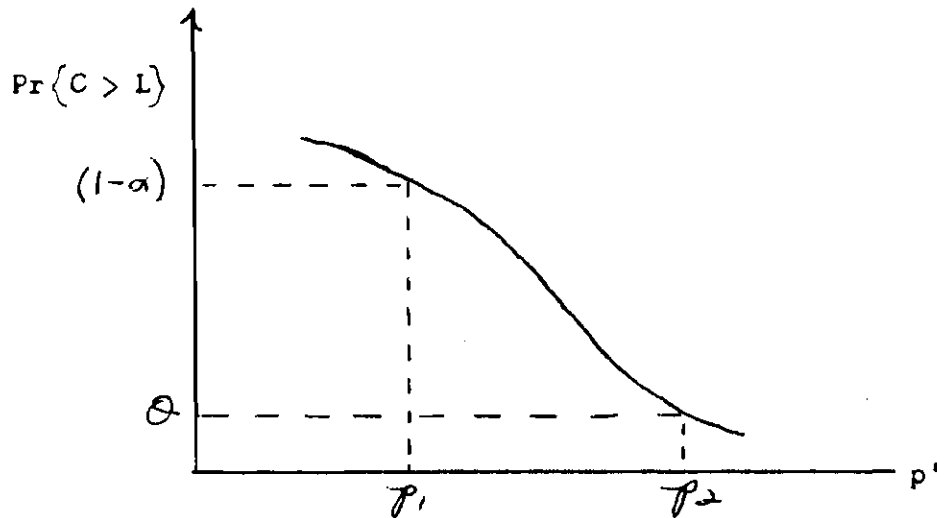


Figure 8. Producer's Cost Characteristic Curve.

Referring once again to the producer's cost characteristic curve, the second point would enable the consumer to penalize the producer for bad quality. If θ had a small value such as 0.05 or 0.10, then the producer would be discouraged from producing bad quality because he would have little likelihood of receiving his minimum payment L if quality was bad. The second point (p_2, θ) has the following probability statement associated with it,

$$(17) \quad \Pr\{C > L \mid p' = p_2\} \leq \theta$$

When quality is good, $p' = p_1$, the producer has high assurance that $C > L$.

$$\Pr\{C > L \mid p' = p_1\} \geq 1 - \alpha$$

The next step in developing a sequential sampling plan with price adjusting is to develop a sequential probability ratio. Following the procedure developed by Wald,

$$SPR = \frac{\Pr(2)}{\Pr(1)}$$

where,

$\Pr (2)$ = probability of getting a value of $C > L$

if H_a is true ($p' = p_2$)

$\Pr (1)$ = probability of getting a value of $C > L$

if H_0 is true ($p' = p_1$)

$\Pr (2)$ can be evaluated as follows,

$$(18) \quad \Pr (2) = \Pr \left\{ C > L \mid p' = p_2 \right\}$$

Using the expression for C , Equation (8) this becomes,

$$\Pr (2) = \Pr \left\{ P \left(\frac{1 - x/n}{1 - p'} \right) > L \mid p' = p_2 \right\}$$

Substituting p_2 for p' and rearranging terms, this becomes,

$$\Pr (2) = \Pr \left\{ x/n < \left[1 - \frac{L}{P} (1 - p_2) \right] \right\}$$

Since the parameters L, P , and p_2 have predetermined values, we can set

$$k_2 = \left[1 - \frac{L}{P} (1 - p_2) \right]$$

hence,

$$(19) \quad \Pr(2) = \Pr \left\{ x < k_2 n \right\}$$

Similarly,

$$\Pr(1) = \Pr \left\{ C > L \mid p' = p_1 \right\}$$

which becomes

$$(20) \quad \Pr(1) = \Pr \left\{ x < k_1 n \right\}$$

where

$$k_1 = \left[1 - \frac{L}{P} (1 - p_1) \right]$$

Now that the SPR has been evaluated as

$$(21) \quad \text{SPR} = \frac{\Pr \{ x < k_2 n \}}{\Pr \{ x < k_1 n \}}$$

the boundaries on the SPR must now be determined. Comparing Figures 7 and 8 we see that the boundaries are:

$$(22) \quad \frac{\theta}{1 - \alpha} \leq \frac{\Pr \{ x < k_2 n \}}{\Pr \{ x < k_1 n \}} \leq \frac{1 - \theta}{\alpha}$$

Since it was assumed that x has a binomial probability distribution, (22) can be rewritten as,

$$(23) \quad \frac{\theta}{1 - \alpha} \leq \frac{\sum_{x=0}^{[k_2 n]} p_2^x (1 - p_2)^{n-x}}{\sum_{x=0}^{[k_1 n]} p_1^x (1 - p_1)^{n-x}} \leq \frac{1 - \theta}{\alpha}$$

where $[k_2 n]$ means the largest integer contained within $k_2 n$.

An analysis of (23) reveals that the sequential probability ratio depends in this case only on the value of n . All one has to do is to solve relationship (23) for n prior to any testing. Take that number of samples, count the number of defectives observed in the sample

and pay $NP(1 - x/n)$ for the lot. Hence, the techniques of sequential analysis do not apply to this type of problem.

Looking back at Foster's single sampling plan, we can see in retrospect why sequential sampling does not apply. Foster's plan specifies only one point on an OC curve, $(p_1, 1 - \alpha)$ and eventually only one variable n need be evaluated.

Foster's plan specifies

$$\Pr \left\{ C > L \mid p^* \leq p_1 \right\} \geq 1 - \alpha$$

which reduces to

$$(24) \quad \Pr \left\{ x/n < k_1 \right\} \geq 1 - \alpha$$

where $k_1 = \left[1 - \frac{L}{P} (1 - p_1) \right]$

Since k_1 can not be less than p_1 (because both L/P and p_1 are less than one) all that is required to insure that $x/n < k_1$ is to find an n large enough to satisfy (24).

In a conventional single sampling plan a reasonable set of stopping rules might be:

- a. Stop if the number of defectives exceeds c ,
the maximum allowable
- b. Or, assuming that $x \leq c$ stop when n observations have
been taken.

With price adjusting sampling plans, the only stopping rule is stop when n observations have been taken. The number of defectives found after n observations is not a factor in the decision to stop the inspection procedure.

Before leaving the subject of sequential analysis, it might be interesting to look at Figure 8 again.

Not only does the concept of sequential analysis not apply for the reasons outlined above, but the consumer can not effectively penalize the producer for submitting bad quality, $p' = p_2$. Recalling Equation (17)

$$\Pr \left\{ C > L \mid p' = p_2 \right\} \leq \theta$$

it is relatively easy to show that θ can never have a value less than 0.5, and that for common values of p_2 such as 0.20 or less, θ is on the order of 0.80. Values of θ which are not significantly smaller than $(1 - \alpha)$ make the idea of a penalty almost meaningless.

Now let us see why θ can never be less than 0.5. For the binomial probability distribution, we know that,

$$E[x] = np'$$

which means, by definition of expected value that

$$\Pr \left\{ x \leq np' \right\} = 0.5$$

Referring to Expression (24) we see that

$$\begin{aligned} \Pr \left\{ C > L \right\} &= \Pr \left\{ x/n < \left[1 - \frac{L}{p} (1 - p') \right] \right\} \\ &= \Pr \left\{ x/n < k \right\} \text{ where } k = \left[1 - \frac{L}{p} (1 - p') \right] \end{aligned}$$

In order for this expression to take on a small value such as 0.05 or 0.10, which is on the order of what we would like θ to be, kn will have to be smaller than np' . This can be seen more readily from

the sketch of the binomial probability distribution shown in Figure 9. The sketch shows $f(x)$ to be continuous for purposes of clarity of concept, although it is actually discrete.

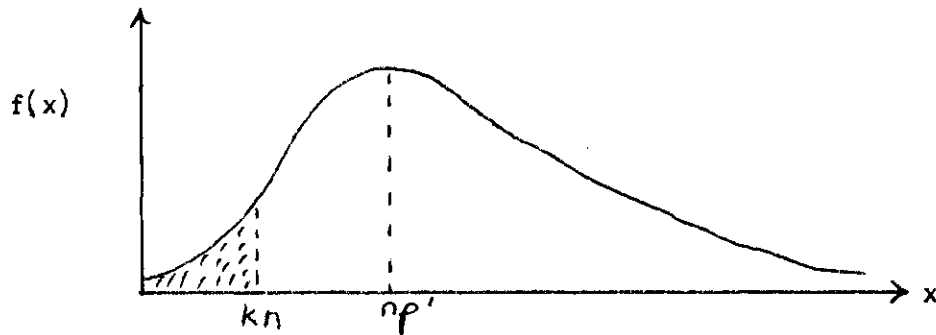


Figure 9. Binomial Probability Distribution

We will show that this is not possible by assuming $kn < np'$ and by showing that this leads to an impossible situation.

$$kn < np'$$

$$\therefore k < p'$$

$$1 - \frac{L}{p} (1 - p') < p'$$

$$1 - \frac{L}{p} + \frac{L}{p} p' < p'$$

$$(1 - \frac{L}{p}) < p'(1 - \frac{L}{p})$$

$$1 < p'$$

The fraction defective can not be greater than one, hence, the assumption that $kn < np'$ must be false.

Barnard's Approach to Sequential Sampling

A discussion of sequential sampling would not be complete without

briefly reviewing Barnard's method which was developed independent of Wald's efforts albeit at about the same time (1946). His method seems particularly appealing since it includes a double sided alternative hypothesis, a concept which sounds like it might accommodate the producer and consumer protection problems simultaneously. Barnard's test determines whether a mean value is significantly more, significantly less or has experienced no important variation from a specified value. A test can be obtained by superimposing two single-sided tests. The procedure consists of plotting the cumulative sum of the observations, taken with μ_0 as the origin, on a chart on which both sets of boundary lines have been drawn. This test does not differ from Wald's sequential test in that the results obtained from each observation may be sufficient to allow testing to be terminated. To satisfy the requirements of the price adjusting plans, one is not concerned with making an evaluation after each observation, only that sufficient observations be made to insure that the confidence limits can be met.

Summary

It has been demonstrated in this chapter that although conventional sequential sampling acceptance plans are more efficient than conventional single sampling plans, a parallel situation does not exist with regard to price adjusting plans. One need only solve for a value of n to satisfy the producer's requirement for protection, and without a need for a limit on the magnitude of x the techniques of sequential sampling simply do not apply. Finally, it was shown that even if a sequential sampling price adjusting plan were possible, it would not be

a powerful plan since θ can never be less than 0.5 and for common values of p_2 it is very close to the value of $1 - \alpha$.

CHAPTER IV

AN ECONOMIC ANALYSIS OF PRICE ADJUSTING SAMPLING PLANS

The purpose of this chapter is to extend the work of Foster to include an economic analysis of a price adjusting single sampling plan. Included will be a development of the limits for the most likely value of C , the variable price per good unit. Also included will be a discussion of the loss functions for both the producer and the consumer which would result from the employment of a price adjusting plan.

Expected Cost Interval

Before either the producer or the consumer should agree to the use of a price adjusting plan, they should have some appreciation for the range of values that C might assume. Conceivably, if x/n is a very bad estimate of the true lot fraction defective, then either the consumer or the producer is going to suffer a potentially severe economic loss. However, if it is known prior to the signing of a final agreement that, for example, 99% of the time C will vary within very tight limits, then both the consumer and producer would be more willing to employ a price adjusting sampling plan.

Specifically, in the following development, we will attempt to determine the areas A_1 and A_2 shown in Figure 10 for given values of P , L , and U .

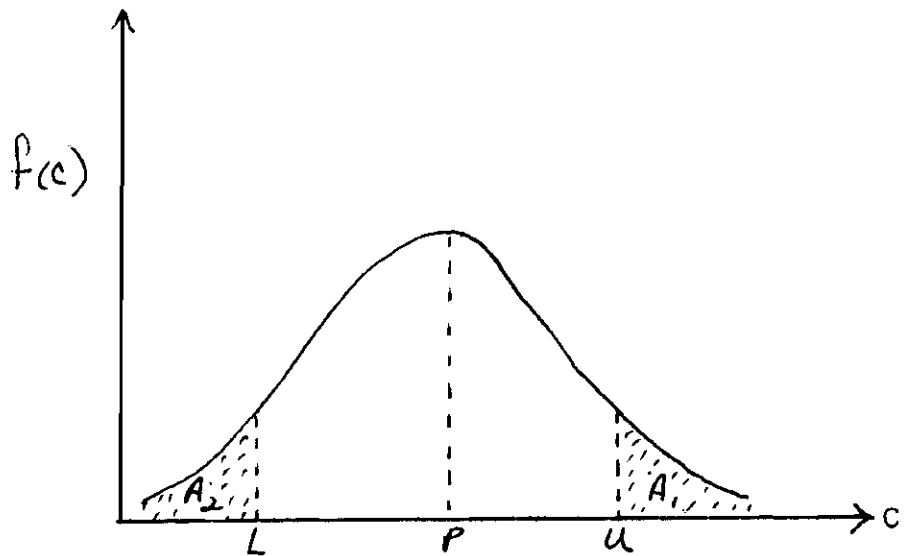


Figure 10. Probability Distribution of C.

To find the expected cost interval, we first need to determine

$$(25) \quad A_1 = \Pr \{ C > U \}$$

and

$$(26) \quad A_2 = \Pr \{ C < L \}$$

so that the interval can be expressed as,

$$(27) \quad \Pr \{ L < C < U \} = 1 - A_1 - A_2$$

The development will in general follow the logic used by Hald (14) in his preliminary work leading to the development of the compound hypergeometric distribution.

Looking first at (25), we recall that $A_1 = \Pr \{ C > U \}$ this means that

$$(28) \quad A_1 = \Pr \{ x < nk_1(p') \}$$

where

$$k_1(p') = \left[1 - \frac{U}{P} (1 - p') \right]$$

Expression (28) is the cumulative distribution function (cdf) of x . To find an expression for the cdf, Hald in his development first derives the marginal distribution of x , i.e., the over-all or average probability of x defectives in the sample as,

$$(29) \quad g_n(x) = \sum_X p(X, x) = \sum_X p\left\{x | X\right\} f_N(X)$$

where

$p\left\{X, x\right\}$ is the simultaneous distribution of X defectives in the lot and x in the sample, $f_N(X)$ is the probability distribution of X defectives in a lot of size N ,

and

$p\left\{x | X\right\}$ is the probability of getting x defectives in the sample given that there are X in the lot.

In the problem with which we are concerned, it is more logical to think of the prior distribution of p' , the true lot fraction defective, than of X , the actual number of defectives in the lot. Therefore, we shall rewrite (29) substituting p' for X ; this substitution produces,

$$(30) \quad g_n(x) = \sum_{p'} p(p, x) = \sum_{p'} p\left\{x | p'\right\} f_N(p')$$

The cumulative distribution may now be written as,

$$(31) \quad A_1 = \Pr\left\{x < nk_1(p')\right\} = \sum_{x=0}^{[nk_1(p')]} \dots$$

$$[nk_1(p')] = \sum_{x=0}^{\infty} \sum_{p'} p^x (1-p')^{n-x} f_N(p')$$

At this point, we shall modify Hald's development further by making the usual assumption that n/N is sufficiently small that the distribution of x may be adequately described by the binomial probability distribution.

Hence, (31) becomes,

$$(32) \quad A_1 = \Pr \left\{ x < nk_1(p') \right\} = \sum_{x=0}^{[nk_1(p')]} \sum_{p'} \left[\binom{n}{x} p'^x (1-p')^{n-x} f_N(p') \right]$$

Similarly, Expression (26) becomes,

$$(33) \quad A_2 = \Pr \left\{ x > nk_2(p') \right\} = 1 - \Pr \left\{ x < nk_2(p') \right\} \\ = 1 - \sum_{x=0}^{[nk_2(p')]} \sum_{p'} \left[\binom{n}{x} p'^x (1-p')^{n-x} f_N(p') \right]$$

where

$$k_2(p') = \left[1 - \frac{1}{p'}(1-p') \right]$$

Example

To illustrate the concept of this section, it will be applied to the example used in Chapter III to illustrate the producer protection plan. The specifications for this problem are,

$$U = 0.225$$

$$P = 0.2083$$

$$L = 0.18$$

$$p_1 = 0.04$$

$$p_2 = 0.08$$

These specifications require that a sample of size $n = 18$ be taken from the lot. One additional assumption will have to be made regarding the nature of $f_N(p')$. Let us assume that when the process is in control $p' = 0.04$ and that based on historical data, this has been the case 80% of the time. We shall use the symbol w_i , $i = 1, 2$, to denote a weighting factor; hence, $w_1 = 0.80$. When the process is out of control, $p' = 0.08$ and $w_2 = 0.20$.

First, we shall solve for A_1 , using Equation (32).

$$\begin{aligned} A_1 &= \sum_{x=0}^{[nk_1(p')]} \sum_{p'} \left[\binom{n}{x} p'^x (1-p')^{n-x} f_N(p') \right] \\ &= \sum_{x=0}^{[nk_1(p')]} \left\{ \left[\binom{n}{x} p_1^x (1-p_1)^{n-x} \right] f_N(p_1) + \right. \\ &\quad \left. \left[\binom{n}{x} p_2^x (1-p_2)^{n-x} \right] f_N(p_2) \right\} \end{aligned}$$

where,

$$f_N(p_1) = w_1 = 0.80$$

$$f_N(p_2) = w_2 = 0.20$$

and, from Expression (28),

$$\begin{aligned}
k_1(p_1) &= \left[1 - \frac{U}{P} (1 - p_1)\right] \\
&= \left[1 - \frac{0.225}{0.2083} (1 - 0.04)\right] \\
&= -0.0368 \\
k_1(p_2) &= \left[1 - \frac{0.225}{0.2083} (1 - 0.08)\right] \\
&= 0.0064
\end{aligned}$$

We note that $k_1(p_1)$ is negative. If we write A_1 in a more simplified form, it will be easier to see the cause and meaning of this negative value.

$$A_1 = \Pr \left\{ x/n < k_1(p_1) \right\} \cdot \Pr \left\{ p' = p_1 \right\} + \Pr \left\{ x/n < k_1(p_2) \right\} \cdot \Pr \left\{ p' = p_2 \right\}$$

The value for $k_1(p_1)$ is negative because the producer's choice of U is so large and p_1 so small that $\frac{U}{P} (1 - p_1) > 1$. The meaning of this result is that $\Pr \left\{ x/n < k_1(p_1) \right\} = 0$ since the probability is zero that the fraction defective will be negative. If U was assigned a value such that $U < \frac{P}{(1 - p_1)}$ then $k_1(p_1)$ would be positive and $\Pr \left\{ x/n < k_1(p_1) \right\}$ would be non-zero.

If the consumer chooses a value of U such that

$$U > \frac{P}{(1 - p_2)}$$

then $k_1(p_2)$ will be negative, and of necessity $k_1(p_1)$ will also be negative.

Note that since

$$p_2 > p_1$$

$$\frac{P}{(1-p_2)} < \frac{P}{(1-p_1)}$$

and if $U < \frac{P}{(1-p_2)}$, then U must also be less than $\frac{P}{(1-p_1)}$.

Hence, if $k_1(p_2) < 0$ then $k_1(p_1) < 0$.

If $k_1(p_2)$ is negative, then $A_1 = 0$.

Hence from Equation (25),

$$\Pr\{C > U\} = 0.$$

which means the consumer is going to have perfect protection no matter what value of n is dictated by a producer protection plan. This condition is discussed in more detail in Chapter VI.

Returning to the example, since $k_1(p_1) < 0$,

$$\Pr\{x/n < 0\} = 0$$

Therefore,

$$\begin{aligned} A_1 &= \Pr\{x/n < k_1(p_2)\} \cdot \Pr\{p' = p_2\} \\ &= \Pr\{x < (18)(0.0064)\} \cdot (0.20) \\ &= 0.0446 \end{aligned}$$

Using Equation (33) we will now evaluate A_2 in a similar manner to that used above.

$$\begin{aligned} A_2 &= 1 - \sum_{x=0}^{[nk_2(p_1)]} [(18 \choose x) (0.04)^x (0.96)^{18-x}] \quad (0.80) - \\ &\quad \sum_{x=0}^{[nk_2(p_2)]} [(18 \choose x) (0.08)^x (0.92)^{18-x}] \quad (0.20) \end{aligned}$$

where

$$\begin{aligned}
 k_2(p_1) &= \left[1 - \frac{1}{P} (1 - p_1) \right] \\
 &= \left[1 - \frac{0.18}{0.2083} (1 - 0.04) \right] \\
 &= 0.171
 \end{aligned}$$

and,

$$\begin{aligned}
 k_2(p_2) &= \left[1 - \frac{0.18}{0.2083} (1 - 0.08) \right] \\
 &= 0.205
 \end{aligned}$$

Hence,

$$\begin{aligned}
 A_2 &= \Pr \left\{ x/n \geq k_2(p_1) \right\} \cdot \Pr(p' = p_1) + \Pr \left\{ x/n \geq k_2(p_2) \right\} \cdot \Pr(p' = p_2) \\
 &= \Pr \left\{ x/n \geq (18)(0.171) \right\} (0.80) + \Pr \left\{ x \geq (18)(0.205) \right\} (0.20) \\
 &= (0.005)(0.80) + (0.05059)(0.20) \\
 &= 0.014
 \end{aligned}$$

Consequently, an evaluation of Equation (27) becomes

$$\begin{aligned}
 \Pr \left\{ 0.18 < C < 0.225 \right\} &= 1 - 0.0446 - 0.014 \\
 &= 0.94
 \end{aligned}$$

This tells the producer and the consumer that for the parameters given at the beginning of this example, if a sample of 18 items is inspected, that 94% of the time the actual price paid per good unit will be in the range \$0.18 to \$0.225, which are limits roughly plus and minus 10% of the desired value, P .

In a similar manner, a second expected cost interval could be developed using the sample size specified for the consumer protection plan. Since in that plan $n = 36$, one would expect that $\Pr\{L < C < U\}$ would be greater than for the plan with $n = 18$, and it is. The procedure is nearly identical to that already presented so only the results will be given. If the consumer protection plan is used,

$$\Pr\{0.18 < C < 0.225\} = 0.9936$$

Producer and Consumer Loss Functions

Linked directly to the spectrum of values that C is likely to assume is the concept of a producer's and a consumer's loss function. The losses, as will be shown in the following development, depend directly on the accuracy with which x/n estimates p^* the true lot fraction defective. The producer will lose money if it happens that $C < P$, i.e., if

$$C = P\left(\frac{1 - x/n}{1 - p^*}\right) < P$$

$$\frac{1 - x/n}{1 - p^*} < 1$$

$$1 - x/n < 1 - p^*$$

or if,

$$x/n > p^*$$

The producer expects that if the plan works perfectly, i.e., $x/n = p^*$, he will theoretically receive for his lot of N items the following total payment,

$$P_{\text{Total}} = NP(1 - p^*)$$

However, since payment is made on the basis of a sampling plan he will actually receive,

$$P_{\text{Total}} = NP(1 - x/n)$$

The producer has already specified the lower limit L on C that he is willing to tolerate, but to enhance his appreciation of the impact that employing a price adjusting plan will have on his business, he should have some appreciation for the total loss on the lot that he may have to absorb. An expression for the producer's loss function can be developed as follows.

$$\begin{aligned} \text{Loss} &= \text{Theoretical } P_{\text{Total}} - \text{Actual } P_{\text{Total}} \\ &= NP(1 - p') - NP(1 - x/n) \\ &= NP(1 - p' - 1 + x/n) \\ &= NP(x/n - p') \end{aligned}$$

Consistent with the earlier discussions which were concerned with protecting the producer against losses per good unit that would occur when $C < L$, the producer's loss function will consider only positive losses. That is, we will assume that the producer is not as concerned with the magnitude of possible profits (negative losses) as he is with the likelihood of potentially ruinous losses. This means that we can write

$$\text{LOSS} = \begin{cases} NP(x/n - p') & x/n > p' \\ 0 & \text{otherwise} \end{cases}$$

We know from the development in Chapter III that the producer's

protection is stated as follows,

$$\Pr \left\{ C < L \mid p' \leq p_1 \right\} \leq \alpha$$

This can be translated into a loss function at quality level $p' = p_1$ as follows,

$$(34) \quad \Pr \left\{ \text{Loss/good unit} > P-L \mid p' = p_1 \right\} < \alpha$$

and

$$(35) \quad \Pr \left\{ \text{Total Lot Loss} > (P-L)N(1-p_1) \right\} \leq \alpha$$

We note that Expression (35) actually represents the total loss on good units in the lot because the loss per good unit is multiplied by the total number of good units in the lot when $p' = p_1$. This can be equated to total lot loss because there is no loss associated with defective items since nothing is paid for them.

Using the data from previous examples in this chapter and Chapter III, we have

$$\alpha = 0.01$$

$$P = \$0.2083$$

$$L = \$0.18$$

$$\text{Assume a lot size, } N = 1,000$$

$$p_1 = 0.04, \quad w_1 = 0.80$$

$$p_2 = 0.08, \quad w_2 = 0.20$$

Therefore, Expression (35) would be evaluated as,

$$\Pr \left\{ \text{Total Lot Loss} > (0.2083 - 0.18)(1,000)(1 - 0.04) \right\} \leq 0.01$$

$$\Pr \left\{ \text{Total Lot Loss} > \$27.17 \right\} \leq 0.01$$

In the long run, the producer can expect to receive an amount for the lot based on the expected value of x/n .

From the prior distribution of p' , we get

$$E[p'] = p_1 w_1 + p_2 w_2 = (0.04)(0.80) + (0.08)(0.20) = 0.048$$

$$\begin{aligned} P_{\text{Total}} &= NP(1 - E[p']) \\ &= 1,000 \$0.2083(1 - 0.048) \\ &= \$198.30 \end{aligned}$$

Hence, based upon the expected total price for the lot, there is no more than a 1% chance of losing more than $\frac{\$27.17}{\$198.30}$ or 13.7% on any transaction; in fact, with a sample size of 18, the risk of a 13.7% loss is actually 0.5%.

A similar analysis of a consumer protection plan can be developed. Define consumer loss as,

$$(36) \quad \text{LOSS} = \begin{cases} NP(p' - x/n), & p' > x/n \\ 0 & \text{otherwise} \end{cases}$$

Using the consumer protection plan developed in Chapter III we have

$$\Pr \left\{ C > U \mid p' = p_2 \right\} \leq \beta$$

The loss per good unit is then,

$$(37) \quad \Pr \left\{ \text{Loss/Good Unit} > U - P \mid p' = p_2 \right\} \leq \beta$$

and the total lot loss is

$$(38) \quad \Pr \left\{ \text{Total Lot Loss} > (U - P)(N)(1 - p_2) \right\} \leq \beta$$

Using the same data with $U = 0.225$, we get

$$\Pr \left\{ \text{Total Lot Loss} > (0.225 - 0.2083)(1,000)(1 - 0.08) \right\} \leq 0.05$$

$$\Pr \left\{ \text{Total Lot Loss} > \$15.36 \right\} \leq 0.05$$

This means that there is no more than a 5% chance that the consumer will lose (i.e., pay an excessive price) more than \$15.36/\$198.30 or 7.7% on any transaction.

Impact of the Producer Protection Plan on the Consumer

In Chapter III a PASS plan was designed which protected the producer. The plan stipulated that a sample of size n be taken from the lot and $NP(1 - x/n)$ be paid for the lot. The statement is made in that chapter that the consumer is agreeable to the plan because he (1) agreed to the indifference curve and (2) he agreed to accept a plan intended to protect the producer. However, he might want to inspect more than n items particularly if as a result of such a plan he suffers potential losses by not having $\Pr \left\{ C > U \mid p' = p_2 \right\}$ be less than β the amount he desires. Of course, if as a result of the producer protection plan, the consumer already is protected with regard to the upper limit on C , then he would have no need to sample further. Let us assume that $\Pr \left\{ C > U \mid p' = p_2 \right\}$ is not less than β . Further, let us assume that the consumer must absorb a cost of sampling C_s , which may be represented by

$$(39) \quad C_s = c_0 + c_1 n$$

That is, there is a fixed set-up cost, perhaps the time required by the inspector to set up his testing equipment, and in addition a cost per

item inspected. If by inspecting more than the prescribed number, n , the consumer can more than recoup his inspection costs by reducing the expected losses associated with $\Pr\{C > U \mid p^* = p_2\}$, then it would be prudent for him to do so. The expression which represents his savings is,

$$\text{Savings} = (\text{Loss if } n = n_0) \cdot \Pr\left\{\text{Loss} \mid n = n_0\right\} \\ - (\text{Loss if } n = n_1) \cdot \Pr\left\{\text{Loss} \mid n = n_1\right\}$$

where $n_1 > n_0$.

We note from the development of Expression (38) that the magnitude of the loss, as defined, is independent of n . Hence, savings becomes,

$$(40) \quad \text{Savings} = (\text{Loss}) \left\{ \Pr(\text{Loss} \mid n = n_0) - \Pr(\text{Loss} \mid n = n_1) \right\}$$

If Savings is greater than $c_1(n_1 - n_0)$ then the additional inspection should be performed. Note that the fixed cost, c_0 , drops out since once the inspector is set up, the only additional cost associated with inspecting a few more items is the variable cost.

Example

In the example of the producer protection plan in Chapter III, the parameters of the plan dictate that $n = 18$. We know from an evaluation of Expression (38) that we are concerned with the total lot loss being greater than $(U - P)(N)(1 - p_2)$ which, when evaluated is \$15.36. Next, we must evaluate

$$\Pr\left\{\text{Loss} \mid n = n_0\right\} \quad \text{where } n_0 = 18$$

Referring to the development of (38) we see that

$$\Pr \left\{ \text{Loss} \mid n = n_0 \right\} = \Pr \left\{ C > U \mid p' = p_2, n = 18 \right\}$$

from expression (13) this becomes,

$$= \Pr \left\{ x/n > 1 - U/P(1 - p_2) \right\}$$

evaluating, using the same parameters as in Chapter III, we get

$$\begin{aligned} &= \Pr \left\{ x > (18)(0.0064) \right\} \\ &= \Pr \left\{ x > 0.115 \mid n = 18, p' = 0.08 \right\} \\ &= 0.77706 \end{aligned}$$

The next task is to search the cumulative tables of the binomial distribution for a minimum value of n which will satisfy the consumer's risk without violating the producer's protection. For this example $n = 36$, which is the solution to the consumer protection problem of Chapter III, is sufficiently large that it more than guarantees the desired producer protection. Hence,

$$\begin{aligned} \Pr \left\{ \text{Loss} \mid n = 36 \right\} &= \Pr \left\{ x < (36)(0.0064) \right\} \\ &= 0.0497 \end{aligned}$$

Therefore evaluating Savings we get,

$$\begin{aligned} \text{Savings} &= \$15,36(0.77706 - 0.0497) \\ &= \$11.15 \end{aligned}$$

If \$11.15 is less than $c_1(36 - 18)$, such as would be the case if $c_1 = \$1.00$ per item, then the additional samples should not be taken. The consumer cannot reap any additional benefits by further sampling. However, if $18 c_1 < \$11.15$, the consumer should inspect an additional 18 items and enjoy additional savings.

CHAPTER V

EFFECT OF TIGHTENED LINEAR INDIFFERENCE

Early in Chapter III it was mentioned that the linear indifference curve could be the curve shown in Figure 11.

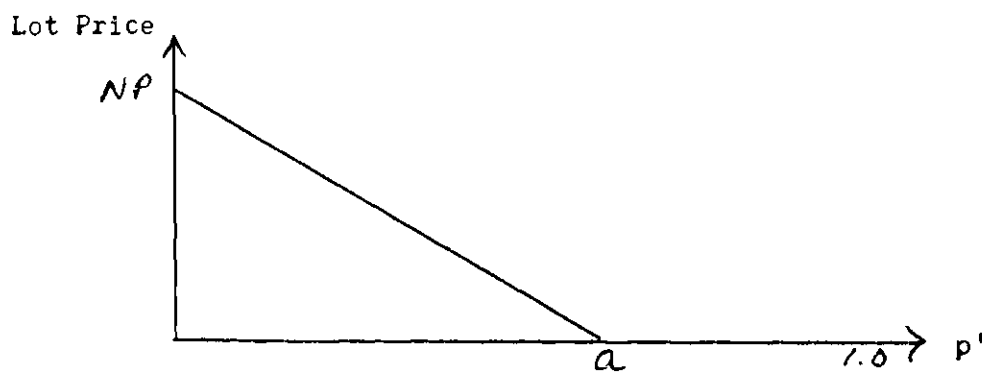


Figure 11. Linear Indifference Curve ($a < 1$).

This curve is described by the equation

$$(41) \quad \text{Lot Price} = \begin{cases} NP(1 - p'/a) & 0 \leq p' < a, \quad 0 < a \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

All of the development up to this point has assumed that $a = 1.0$, meaning that the indifference curve intercepts the abscissa at $p' = 1.0$. With $0 < a < 1$ the pricing scheme is dramatically altered. With this new definition for Lot Price, the new expression for P_{Total} becomes,

$$(42) \quad P_{\text{Total}}^* = \begin{cases} NP[1 - x/an], & (1 - x/an) > 0, \quad 0 < a \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

The development of (42) will be omitted since it parallels very closely

the evolution of Equation (7) from Equation (2).

The practical meaning of (42) is that so long as $x/n < a$, the consumer will accept the lot and pay an amount less than would be paid using the plan developed earlier, but if $x/n \geq a$, then the consumer will accept the lot but pay nothing for it. The difference between the amount paid for the lot under both plans can be readily evaluated.

$$(43) \quad \begin{aligned} \text{Difference} &= NP(1 - x/n) - NP(1 - x/an) \\ &= (NP \ x/an)(1 - a) \end{aligned}$$

Using this kind of tightened indifference curve, the consumer would first identify p_1 , p_2 , and then choose \underline{a} such that $p_2 < a \leq 1$.

The introduction of a value for \underline{a} less than 1.0 not only affects the total price per lot, it also affects C , the actual price paid per good unit.

Recall that,

$$C^* = \frac{\text{Total Price paid for the Lot}}{\text{Actual number of good units in the lot}}$$

We use the double-prime notation to differentiate between the two expressions for actual price per good unit.

$$(44) \quad \begin{aligned} C'' &= \frac{NP(1 - x/an)}{N(1 - p')} \\ C'' &= P \left(\frac{1 - x/an}{1 - p'} \right) \end{aligned}$$

In the earlier analysis, $E[C] = P$. However, now,

$$E[C] = P \left(\frac{1 - p'/a}{1 - p'} \right) = (P/A) \left(\frac{a - p'}{1 - p'} \right)$$

which means that as p' increases toward a , the producer can expect not only to receive less for the total lot, but he can also expect to be penalized further for bad quality in that he actually receives less per good unit. When,

$$\begin{aligned} p' &= a \\ E[C] &= 0 \end{aligned}$$

Although having $a < 1$ acts to severely penalize the producer when p' approaches a , the parameters of the problem can be modified so that the producer will have an incentive to produce lots with $p' < p_1$. As a starting point, a new value for price per good unit will have to be chosen such that the expected value of the total lot price when $p' = p_1$, will be equal to the total amount the producer would normally receive using a conventional plan. This concept can be best illustrated by working an example. The following parameters will be used,

$$\begin{aligned} N &= 1,000 \\ p_1 &= 0.04 \\ p_2 &= 0.08 \\ a &= 0.70 \end{aligned}$$

$$\begin{aligned} \text{Basic price per unit using a conventional plan} \\ &= \$0.20 \text{ per each unit} \end{aligned}$$

We shall first solve first solve for P'' the new price per good unit.

$$\begin{aligned} \text{Expected total lot price under conventional plan} &= \$0.20N \\ &= \$200.00 \end{aligned}$$

$$\begin{aligned}
 E[P_{\text{Total}}^n \mid p' = p_1] &= NP^n(1 - p_1/a) \\
 &= 1,000P^n(1 - \frac{0.04}{0.70}) \\
 &= 943P^n
 \end{aligned}$$

Equating the two expressions for lot price evaluated at the base point p_1 , we get,

$$p^n = \frac{200}{943} = \$0.2121$$

Note the following schedule of payments which clearly shows the incentive for good quality and penalty for bad.

$$E[P_{\text{Total}}^n \mid p' = 0] = \$212.10 \quad (6\% \text{ over base price})$$

$$\begin{aligned}
 E[P_{\text{Total}}^n \mid p' = 0.04] &= \$200.00 \quad (\text{base price -} \\
 &\quad \text{also same value as for earlier PASP plan} \\
 &\quad \text{with } a = 1.0)
 \end{aligned}$$

$$E[P_{\text{Total}}^n \mid p' = 0.08] = \$187.90 \quad (6\% \text{ penalty for bad quality})$$

$$E[P_{\text{Total}}^n \mid p' = a] = 0$$

The next question is how large must the sample size be to provide the producer the protection he desires. Assume,

$$L = \$0.19 \quad (10\% \text{ less than } P^n)$$

$$\alpha = 0.01$$

Using the usual procedure

$$\Pr\left\{C^n < L \mid p' \leq p_1\right\} \leq \alpha$$

C^n is evaluated using Equation (44) and we get,

$$\Pr \left\{ x/n > a \left[1 - \frac{L}{P''} (1 - p_1) \right] \right\} \leq \alpha$$

Evaluating gives,

$$\Pr \left\{ x/n \geq 0.0952 \right\} \leq 0.01$$

which requires $n = 74$.

We note that the tightened indifference curve with $a = 0.70$ has greatly increased the sample size compared with the earlier plan which had $a = 1.0$. In the earlier example however, a value of $L = \$0.18$ was used which is not exactly 10% less than $P = \$0.2083$. To more accurately demonstrate the impact that varying a has on n , a series of tabulated values is shown in Table 1 which uses a constant ratio $L/P'' = 0.90$.

In order to protect the producer when he submits good quality ($p' = p_1$), he should expect to receive the same amount no matter how a varies. Therefore, for each value of a , unique values of P'' and L will have to be computed.

The relationship for computing P'' results from equating the expected payment from a conventional plan and the payment using a plan with $a < 1.0$. Using the available data, $N = \$1,000$, price = \$0.20 for a conventional plan this becomes,

$$\begin{aligned} \text{Expected Total Payment} \\ \text{Using Conventional Plan} &= E[P'' \mid p' = p_1] \end{aligned}$$

$$(\$0.20)(N) = NP''(1 - p_1/a)$$

or,

$$P'' = \frac{\$0.20}{\left[1 - \frac{0.04}{a} \right]}$$

Table 1. Comparison of Plans for $\alpha < 1.0$.

a	p^n	L	$k_1 a$	$n_a = 0.01$	$n_a = 0.025$	$n_a = 0.030$
1.0	0.2083	0.1875	0.1360	30	6	6
0.90	0.2093	0.1884	0.1223	33	25	17
0.85	0.2099	0.1889	0.1156	44	26	26
0.80	0.2105	0.1895	0.1088	56	28	28
0.75	0.2113	0.1902	0.1020	60	40	40
0.70	0.2121	0.1909	0.0952	74	53	42
0.65	0.2131	0.1918	0.0884	102	68	57
0.60	0.2143	0.1929	0.0816	135	86	74
0.55	0.2157	0.1941	0.0748	190	135	122
0.50	0.2174	0.1957	0.0680	280	230	180
0.45	0.2195	0.1976	0.0612	<p>NOTE:</p> $\Pr \left\{ x/n \geq k_1 a \right\} \leq \alpha$ <p>where</p> $k_1 = 1 - \frac{L}{p^n} (1 - p_1)$		
0.40	0.2222	0.2000	0.0544			
0.35	0.2258	0.2032	0.0476			
0.30	0.2308	0.2077	0.0408			

and hence

$$L = 0.90 P''$$

Looking at Table 1, we draw two rather interesting conclusions.

- (1) As the value of (ak_1) in the expression

$$\Pr \left\{ x/n > (ak_1) \right\}$$

approaches p' (which is evaluated at 0.04) the sample size increases without bound. This results from the fact that as (ak_1) approaches p_1 from the positive side, n must tend toward infinity to insure that $\Pr \left\{ x/n > (ak_1) \right\} \leq \alpha$.

- (2) As the value of α is increased, the sample size is generally reduced. The value of n does not always decrease for an incremental increase in α because of the discrete nature of x . It is of interest to note that when $\alpha \geq p_1$, then no matter what values α assumes (as long as $ak_1 > p_1$) then $n = 1$. The proof of this is located in the Appendix.

To answer the obvious question of how the consumer chooses a value for \underline{a} , one approach which will aid the consumer in arriving at a decision involves looking at how the increase in sampling costs are offset by the reduction in the expected payment for the total lot. This analysis can best be demonstrated by working an example. We will use the same 2-point binomial for p' as has been used earlier. Recall that

$$\begin{aligned} \Pr \left\{ p' = p_1 \right\} &= 0.80 \\ \Pr \left\{ p' = p_2 \right\} &= 0.20 \end{aligned}$$

Let the increase in sampling costs equal,

$$\Delta C_s = c_1(n_a < 1.0 - n_a = 1.0)$$

The expected reduction, $R(a)$, can be identified as,

$$R(a) = (\text{Reduction} \mid p' = p_1) \Pr \{p' = p_1\} + \\ (\text{Reduction} \mid p' = p_2) \Pr \{p' = p_2\}$$

The decision will be to reduce a so that $|\Delta C_s - R(a)| < \epsilon$ where ϵ is some tolerable difference necessitated by the discrete nature of the binomial variable.

From the decision to use p_1 , the acceptable quality level, as a pivot point,

$$(\text{Reduction} \mid p' = p_1) = 0$$

therefore, for the 2-point binomial case,

$$R(a) = (\text{Reduction} \mid p' = p_2) \Pr \{p' = p_2\}$$

To find a minimum value for $|\Delta C_s - R(a)|$, a trial and error method will be used.

$$\begin{aligned} (\text{Reduction} \mid p' = p_2) &= E[P_{\text{Total}}^n \mid p' = p_2, a = 1.0] - \\ &E[P_{\text{Total}}^n \mid p' = p_2, a = 0.90] \\ &= (NP_{a=1.0}^n)(1 - \frac{p_2}{1.0}) - (NP_{a=0.90}^n)(1 - \frac{p_2}{0.9}) \end{aligned}$$

The values for P_a^n are obtained from Table 1.

$$\begin{aligned}
 (\text{Reduction} \mid p' = p_2) &= (1,000)(0.2083)(1 - 0.08) - \\
 &\quad (1,000)(0.2093)(1 - \frac{0.08}{0.9}) \\
 &= \$1.94
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 R(a) &= (\$1.94) \Pr \left\{ p' = p_2 \right\} = (\$1.94)(0.20) \\
 &= \$0.39
 \end{aligned}$$

Since $n_{a=0.90} = 33$ and $n_{a=1.0} = 30$, the additional sampling costs are $3c_1$. If $c_1 < \$0.13$ per item, then $a = 0.90$ would be a logical choice based on purely economic considerations. However, other factors which can not be readily quantified may affect the consumer's decision on a value for a . For example, the consumer may experience severe work stoppages if the product he is buying is used on an assembly line and $p' > 0.50$. He may then choose $a = 0.50$ especially if his per item sampling cost, c_1 , is small. Additionally, the consumer may want to absorb the higher sampling costs associated with a smaller value of a if he would like to encourage the producer to improve his production process so as to reduce the lot fraction defective below p_1 . Table 2 shows the total payment for a lot of 1,000 items if $p' = 0$.

In summary, it has been the purpose of this chapter to show that by allowing a to assume values less than one the consumer increases his flexibility in arriving at a payment plan. With $a < 1.0$ the consumer has the capability to more severely penalize the producer for bad quality while simultaneously furnishing him with an incentive to improve the quality of his process. The producer, on the other hand, may or may not

Table 2. Payment Schedule Corresponding
To Decreasing Values of a

<u>a</u>	<u>if $p' = 0$</u>	<u>Bonus*</u>
1.0	\$208.30	4.15%
0.9	209.30	4.65
0.85	209.90	4.95
0.80	210.50	5.25
0.75	211.30	5.65
0.70	212.10	6.05
0.65	213.10	6.51
0.60	214.30	7.15
0.55	215.70	7.85
0.50	217.40	8.70

* Bonus is computed with regard to the base price of \$200.00 paid when $p' = p_1$, the acceptable quality level.

be willing to accept a plan with $a < 1.0$. If he has reason to believe that he can reduce his fraction defective below p_1 , then he would be eager to accept the plan because of the additional revenue it would provide as compared with a plan using $a = 1.0$. However, even though he is assured by the new plan that

$$\Pr \left\{ C'' < L \mid p' = p_1 \right\} \leq \alpha$$

he may hesitate to accept it if he fears his fraction defective may increase beyond p_1 .

CHAPTER VI

SUMMARY AND CONCLUSIONS

Introduction

The purpose of this chapter is to summarize the material presented in the preceding chapters. The advantages and disadvantages of price adjusting sampling plans will be reviewed with particular emphasis placed on the practical problems that this type of plan presents. The price adjusting plan with $\alpha < 1.0$ provides such a unique combination of rewards and penalties for the producer that this type of plan will be recapitulated separately. Finally, recommendations will be made for possible future research dealing with the problem of minimizing the combination of potential losses and the cost of sampling to reduce those losses.

The Producer and Consumer Protection Plans

With both the producer and the consumer protection plans there is a direct correlation between p' and the sample size necessary to provide the desired protection. For example, with regard to the producer protection plan, with a fixed value for $\frac{L}{P}$, n decreases as p' decreases. This results from the fact that a reduction in p' reduces the variance of \bar{x}/n which is equal to $p'(1 - p')/n$. Hence, with a reduced variance fewer observations are necessary to insure the desired protection.

For a fixed value of p' there is also a correlation between the ratio L/P and n .

We can see from Expression (11),

$$\Pr \left\{ x/n \geq 1 - \frac{L}{p} (1 - p') \right\} \leq \alpha,$$

that as $\frac{L}{p} \rightarrow 1$, $[1 - \frac{L}{p} (1 - p')] \rightarrow p'$.

Expression (11) then approaches

$$\Pr \left\{ x/n \geq p' \right\} \leq \alpha.$$

For any value of p'

$$\Pr \left\{ x/n \geq p' \right\} = 0.50.$$

Since α is much less than 0.50, the sample size $n \rightarrow \infty$ as $\frac{L}{p} \rightarrow 1$.

An interesting situation can arise with the consumer protection plan that can not arise with the producer protection plan and that is the case where no observations need be taken. If we look at Expression (13)

$$\Pr \left\{ x/n > 1 - \frac{U}{p}(1 - p_2) \right\} \geq 1 - \beta$$

we see that

$$[1 - \frac{U}{p} (1 - p_2)] \text{ can be non-positive.}$$

If,

$$1 - \frac{U}{p} (1 - p_2) \leq 0$$

it means that if

$$\frac{U}{p} \geq \frac{1}{1 - p_2}$$

then no observation need be taken since,

$$\Pr \left\{ x/n \geq 0 \right\} = 1$$

This may also be interpreted as indicating that the consumer is guaranteed perfect protection.

As outlined by Foster (12) and Foster and Perry (13), there are a number of advantages and disadvantages to price adjusting sampling plans. The advantages include:

- (a) Price adjusting sampling eliminates the costs associated with rejected lots.
- (b) The time lost between rejecting one lot and the reception of its replacement is avoided.
- (c) Since the consumer is indifferent to quality he need not maintain representatives at the producer's facility to insure that good quality is produced, as is often done with government contracts.
- (d) An incentive to produce good quality is built into a price adjusting sampling plan. If $p' = 0$, $P_{\text{Total}} = NP$, whereas if $p' = 1.0$, $P_{\text{Total}} = 0$.

Further incentives are added when a plan with $a < 1.0$ is used, but that will be discussed in more detail later in this chapter.

Price adjusting sampling plans, despite their many advantages have some very real disadvantages. Included among them are,

- (a) The difficulty that would be encountered in trying to convince a producer and consumer to accept a plan which bases

payment on expected value concepts. This also will be discussed in detail in this chapter.

- (b) Since total lot price depends on the number of defectives found in a sample, some difficulty might be encountered in insuring that the inspection results are fair.

Although it could not be considered a disadvantage, certainly it is a disappointment that it isn't possible to combine the economics of item-by-item sequential sampling with the many advantages of price adjusting sampling.

To summarize that analysis, the reason that sequential sampling can not be applied to price adjusting sampling is that to insure

$$\Pr \left\{ C < L \mid p' = p_1 \right\} \leq \alpha$$

or

$$\Pr \left\{ C > U \mid p' = p_2 \right\} \leq \beta$$

requires that x/n , the unbiased estimator of p' , the true fraction defective, be more or less than some fraction. To meet the specification for x/n merely requires that n be of some minimum size, and this minimum size can be calculated prior to taking any observations. Hence, there is no need to determine the number of defectives accumulated after each observation. Ergo, there is no need for sequential sampling.

The Economics of Price Adjusting Sampling Plans

The purpose of Chapter IV was to demonstrate that before the producer and the consumer agree to a price adjusting plan, they can be furnished relatively complete information on the economic risks involved.

The actual implementation of a PASP might work as follows. The consumer would specify N and p_1 , what he considers a good level of quality. Bids submitted by vendors would specify P , L , and α . The consumer would look at each set $\{P, L, \alpha\}$ and compute the sample size n necessary to give each vendor his desired protection. Next, for each value of n , he would determine the level of protection he would receive vis-a-vis U , his upper limit on C . He would then award a contract based on an overall evaluation of the following criteria:

- (a) P , price per good unit
- (b) The sample size necessary to give the producer his desired protection, (recall that variable cost of sampling is $c_1 n$)
- (c) The impact that taking a sample of n would have on the consumer's desired protection.

One of the problems that could arise with a plan that bases payment on the results of a sampling inspection is that either the producer or the consumer or both may need to know what the payment will be prior to the sampling inspection. Suppose for example that Consumer B buys metal parts for his product from Vendor A. Before Consumer B can quote a price on his product he must know what the cost of his metal components will be. This is easily handled if Consumer B and Vendor A expect to continue doing business for more than just one transaction. The price that Consumer B pays is $NP(1 - \hat{p})$ where \hat{p} is based on Vendor A's process average. In the example of Chapter IV this would be

$$\hat{p} = p_1 w_1 + p_2 w_2$$

Should it happen that $NP(1 - p) > NP(1 - x/n)$ then the amount paid for

the next lot (again prior to inspection sampling) is $NP(1 - \hat{p})$ minus the overpayment from the previous transaction. Similarly, if $NP(1 - \hat{p}) > NP(1 - x/n)$ the payment for the next lot would be increased by the amount of the underpayment. Whenever Consumer B decides to stop buying from Vendor A, the minor difference could finally be transferred from one to the other.

Price Adjusting Sampling Plans with $a < 1.0$

As was mentioned in Chapter V, a price adjusting plan with $a < 1.0$ has considerable appeal to the consumer. Recalling that the plan with $a = 1.0$ and the one with $a < 1.0$ were designed so that at $p' = p_1$ expected total lot payments were equal, the schedule of payments shown in Table 3 illustrates the increased incentives and penalties possible when $a < 1.0$.

Table 3. Schedule of Payments

$E[P_{\text{Total}} p' = 0] = \208.30 (4% incentive)	$E[P_{\text{Total}}^* p' = 0] = \212.10 (6% incentive)
$E[P_{\text{Total}} p' = 0.04] = \200.00	$E[P_{\text{Total}}^* p' = 0.04] = \200.00
$E[P_{\text{Total}} p' = 0.08] = \192.00	$E[P_{\text{Total}}^* p' = 0.08] = \187.90
$E[P_{\text{Total}} p' = 0.70] = \125.00	$E[P_{\text{Total}}^* p' = 0.70] = 0$

As was mentioned in the survey of the literature, Chapter II, most acceptance sampling plans only penalize the producer when $p' > p_1$, but provide no incentives to the producer to improve his process in order to reduce lot fraction defective below p_1 . MIL-STD-105D does offer

indirect incentives in that the inspection criteria may be tightened or reduced depending on the pattern of inspection results, but no direct monetary reward or penalty is possible. With price adjusting sampling using a $\alpha < 1.0$, significant rewards and penalties can be easily incorporated in the plan.

Recommendations for Future Research

In the economic analysis of price adjusting sampling plans, there was no attempt made to minimize the losses that the consumer and producer experience when $p \neq p'$. In Chapter III the loss functions were defined as follows,

Producer's Loss Function:

$$\text{Loss} = \begin{cases} NP(x/n - p'), & x/n > p' \\ 0 & \text{otherwise} \end{cases}$$

Consumer's Loss Function:

$$\text{Loss} = \begin{cases} NP(p' - x/n), & p' > x/n \\ 0 & \text{otherwise} \end{cases}$$

From the producer's point of view, the way to minimize losses is to take a sample of size $n = N$. This will insure that $p = p'$. However, if the producer had to share the cost of sampling with the consumer, he would be motivated to find a sample size which minimized the combination of the total expected losses due to estimation and the total cost of sampling. In the latter case, a logical loss function would be,

$$(45) \quad \text{Loss} = NP |p - p'|$$

and sampling costs would be described as,

$$C_s = C_0 + c_1 n$$

Although (45) is a linear loss function, it might well be in the best interests of the producer and the consumer to use a quadratic loss function such as

$$(46) \quad \text{Loss}' = NP(p - p')^2$$

which has the advantage that it reduces the possibility of large losses more effectively than a linear loss function.

The approach outlined in Raiffa and Schlaifer (16) involves finding the estimate p^* which will minimize the expected estimation loss (either linear or quadratic) as defined by either (45) or (46). Once the value of p^* has been found, it will be used to determine first, the magnitude of the expected losses for a given prior distribution and second, the value of n which will minimize the sampling costs.

This approach is quite different from that discussed in this paper and is recommended for future research because it offers an interesting alternative. This approach offers the opportunity to determine a sample size which minimizes losses whereas Foster's approach identifies another value for n which provides the producer, for example, with a high assurance of receiving some minimum payment, L .

It is recommended that future work in the area of price adjusting sampling plans address the problem of optimizing the choice of a , the intercept point on the abscissa for the linear indifference curve. Although intangibles will affect the consumer's choice of a , if the

difference between the linear indifference curve and the consumer's convex ideal indifference curve (see Figure 12) could be interpreted as a loss, then perhaps some relationship could be developed involving the magnitude of that loss, a , the intercept and the total cost of sampling n items.

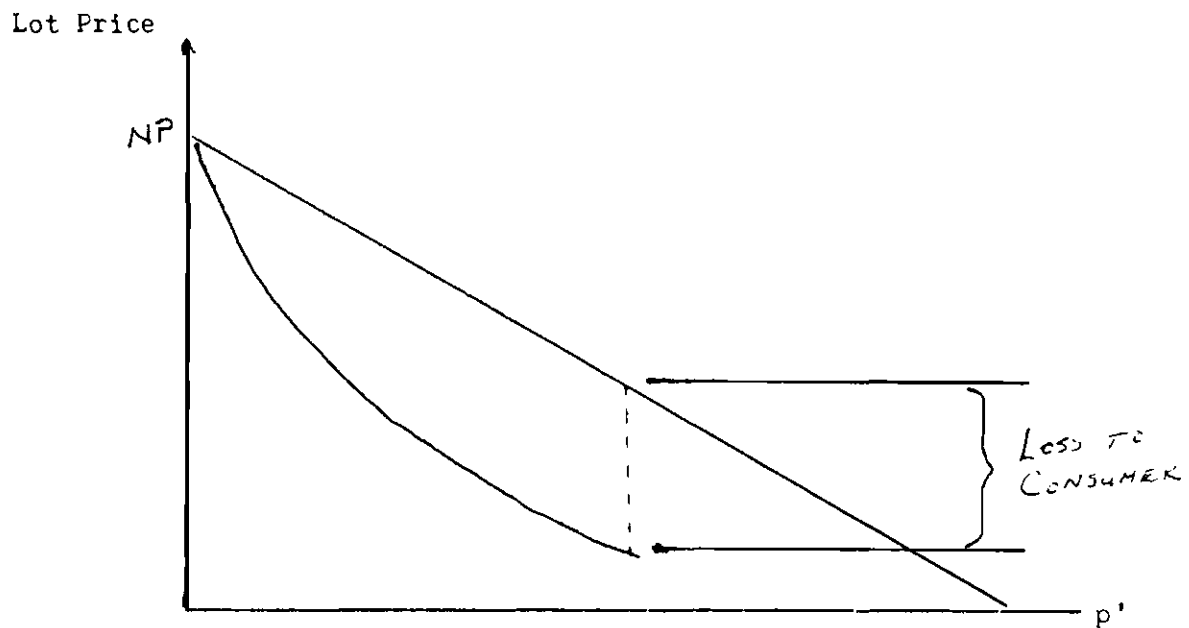


Figure 12. Consumer Indifference Curves.

Similarly, work could be done in the area of minimizing the comparable losses that the producer suffers when he agrees to accept a linear rather than a concave indifference curve.

APPENDIX

Proof that for $\alpha \geq p_1$ the sample size necessary to give the producer his desired protection is, $n = 1$. Let

$$k_1 = \left[1 - \frac{1}{p} (1 - p_1) \right]$$

Since $\frac{1}{p} < 1.0$, $\alpha \leq 1.0$ and $p_1 < 1.0$, then $(\alpha k_1) < 1.0$.

Therefore, the producer protection requirement becomes for $n = 1$,

$$\Pr \left\{ x/n \geq \alpha k_1 \right\} = \Pr \left\{ x \geq \alpha k_1 \right\}$$

Since x is discrete, this becomes

$$\Pr \left\{ x \geq \alpha k_1 \right\} = \Pr \left\{ x \geq [\alpha k_1] + 1 \right\}$$

where $[\alpha k_1]$ means "the largest integer contained in."

Since $\alpha k_1 < 1.0$, $[\alpha k_1] = 0$ and $[\alpha k_1] + 1 = 1$

$$\therefore \Pr \left\{ x \geq 1 \mid n = 1 \quad p' = p_1 \right\} = p_1$$

Therefore, if $p_1 \leq \alpha$, the producer protection requirement can be satisfied with a sample of $n = 1$.

Q.E.D.

The implicit meaning of this proof is that if the producer specifies $\alpha \geq p_1$, then the plan is to take a sample of one, and $(1 - p_1)$ of the time the producer gets a maximum payment, and p_1 of the time he receives zero dollars for the lot. Since $p_1 \leq \alpha$, this should be acceptable to the producer as a long range plan.

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